ASSIGNMENT PROBLEM WITH GENERALIZED INTERVAL ARITHMETIC

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Abstract - Linear assignment problems are very well known linear programming problems of assignment problem. In a ground reality the entries of the cost matrix are not always crisp. In many application this parameters are uncertain and this uncertain parameters are represented by interval. In this contribution we propose a new computational procedure to solve assignment problem with generalized interval arithmetic interval Hungarian method.

Index Terms— Interval Numbers, generalized interval arithmetic, Interval Linear Programming, Ranking

1. INTRODUCTION

In practical field we are sometime faced with type of problem which consists of jobs to machines, drivers to trucks, men to offices etc. in which the assignees possess varying degree of efficiency, called as cost or effectiveness. The basic assumption of this type of problem is that one person can perform one job at a time. An assignment plan is optimal if it minimizes the total cost or maximizes the profit. This type of linear assignment problems can be solved by the very well-known Hungarian method which was derived by the two mathematician Kuhn, H. W [14]. Authors are proposed different methods to handle different types of assignment problems. In this context, Albrecher [1, 10] introduced an asymptotic behaviour of bottleneck problems and Aldous [3] studied asymptotes in the random assignment problem. Alvis and Lai [4] coined out the probabilistic analysis of a heuristic for the assignment problem .Quadratic assignment was analyzed by many authors in different approaches [4, 5, 7, 8]. Burkard [6] proposed time-slot assignment for TDMA systems .Burkard et al. [9] studied an algebraic approach to assignment problems. Pardalos and Pitsoulis [22] developed some works on nonlinear assignment problems. Asymptotic properties of the random assignment problem have been studied by Olin [21]. Nair Proved of the Parisi and Coppersmith-Sorkin conjectures in the random assignment problem [20]. Fitness landscape analysis and memetic algorithms for the quadratic assignment problem are described by Merz and Freisleben [18]. Optimal Permutations and Bottleneck quadratic assignment problems and the bandwidth problem were analyzed by Li et al. and Kellerer et al. [20]. Krokhmal et al. [15] described asymptotic behaviour of the expected optimal value of the multidimensional assignment problem. In the realistic problems costs are not always in a crisp form, sometime these parameters are uncertain which are represented by intervals. Hence we need the help of interval analysis for handling this type of data. In this paper a general interval linear assignment problem is taken into consideration with basis assumption that one person can perform one job at a time. Here the new method has been proposed to handle such type of problem. We solved one example problem using this proposed method. Corresponding results are computed and has been reported here. The rest of the paper is organized as follows: in the next section review on interval arithmetic are highlighted. In section 3 the detail of proposed interval Hungarian method are presented.In section 4 example problems are solved and results are analyzed .Finally conclusions are drawn in section 5.

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2. PRELIMINARIES

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

Let $\tilde{a} = [a_1, a_2] = \{x : a_1 \le x \le a_2, x \in R\}$. If $\tilde{a} = a_1 = a_2 = a$, then $\tilde{a} = [a, a] = a$ is a real number (or a degenerate interval). Let $IR = \{\tilde{a} = [a_1, a_2] : a_1 \le a_2 \text{ and } a_1, a_2 \in R\}$ be the set of all proper intervals and $\overline{IR} = \{\tilde{a} = [a_1, a_2] : a_1 > a_2 \text{ and } a_1, a_2 \in R\}$ be the set of all improper intervals on the real line R. We shall use the terms"interval" and"interval number" interchangeably.

The mid-point and width (or half-width) of an interval number $\tilde{a} = [a_1, a_2]$ are defined as $m(\tilde{a}) = \left(\frac{a_1 + a_2}{2}\right)$ and $w(\tilde{a}) = \left(\frac{a_2 - a_1}{2}\right)$. The interval number $\tilde{a}$ can also be expressed in terms of its midpoint and width as $\tilde{a}=[a_1,a_2]=\langle m(\tilde{a}),w(\tilde{a})\rangle$.

2.1. A New Interval Arithmetic

Ming Ma et al. **Error! Reference source not found.** have proposed a new fuzzy arithmetic based upon both location index and fuzziness index function . The location index number is taken in the ordinary arithmetic, whereas the fuzziness index functions are considered to follow the lattice rule which are the least upper bound and greatest lower bound in the lattice L. That is for $a,b\in L$ we define $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$.

For any two intervals $\tilde{a}=[a_1,\ a_2]$, $\tilde{b}=[b_1,\ b_2]\in IR$ and for $*\in\{+,-,\cdot,\div\}$, the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as:

$$\tilde{a}*\tilde{b}=[a_1,\ a_2]*[b_1,\ b_2]=\ \langle m(\tilde{a}),w(\tilde{a})\rangle*\langle m(\tilde{b}),w(\tilde{b})\rangle$$
$$=\langle m(\tilde{a})*m(\tilde{b}),\ \max\{w(\tilde{a}),w(\tilde{b})\}\rangle.$$

In particular

(i). Addition : $\tilde{a}+\tilde{b}=\ \langle m(\tilde{a}),w(\tilde{a})\rangle+\langle m(\tilde{b}),w(\tilde{b})\rangle$
$$=\langle m(\tilde{a})+m(\tilde{b}),\ \max\{w(\tilde{a}),w(\tilde{b})\}\rangle.$$

(ii). Subtraction : $\tilde{a}-\tilde{b}=\ \langle m(\tilde{a}),w(\tilde{a})\rangle-\langle m(\tilde{b}),w(\tilde{b})\rangle$
$$=\langle m(\tilde{a})-m(\tilde{b}),\ \max\{w(\tilde{a}),w(\tilde{b})\}\rangle.$$

(iii). Multiplication : $\tilde{a}\times\tilde{b}=\ \langle m(\tilde{a}),w(\tilde{a})\rangle\times\langle m(\tilde{b}),w(\tilde{b})\rangle$
$$=\langle m(\tilde{a})\times m(\tilde{b}),\ \max\{w(\tilde{a}),w(\tilde{b})\}\rangle.$$

(iv). Division : $\tilde{a}\div\tilde{b}=\langle m(\tilde{a}),w(\tilde{a})\rangle\div\langle m(\tilde{b}),w(\tilde{b})\rangle$
$$=\langle m(\tilde{a})\div m(\tilde{b}),\max\{w(\tilde{a}),w(\tilde{b})\}\rangle,$$
provided $m(\tilde{b})\not\approx\tilde{0}$.

2.2. Ranking of Interval Numbers

Sengupta and Pal [2] proposed a simple and efficient index for comparing any two intervals on IR through decision maker's satisfaction.

Definition 2.2.1. Let $\preceq$ be an extended order relation between the interval numbers $\tilde{a}=[a_1,\ a_2]$, $\tilde{b}=[b_1,\ b_2]$ in IR, then for $m(\tilde{a})<m(\tilde{b})$, we construct a premise $(\tilde{a}\circ\tilde{b})$ which implies that $\tilde{a}$ is inferior to $\tilde{b}$ (or $\tilde{b}$ is superior to $\tilde{a}$).

An *acceptability function* $A_{\preceq}:IR\times IR\rightarrow[0,\infty)$ is defined as:

$$A_{\preceq}(\tilde{a},\tilde{b})=A(\tilde{a}\preceq\tilde{b})=\frac{m(\tilde{b})-m(\tilde{a})}{w(\tilde{b})+w(\tilde{a})},\text{ where } w(\tilde{b})+w(\tilde{a})\neq0.$$

$A_{\prec}$ may be interpreted as the grade of acceptability of the "the first interval number to be inferior to the second interval number". For any two interval numbers $\tilde{a}$ and $\tilde{b}$ in IR either $A(\tilde{a}\preceq\tilde{b})\geq0$ (or) $A(\tilde{b}\succeq\tilde{a})\ \succeq\ 0$ (or)

$A(\tilde{a}\preceq\tilde{b})=0$ (or) $A(\tilde{b}\succeq\tilde{a})=0$ (or) $A(\tilde{a}\preceq\tilde{b})+A(\tilde{b}\ \preceq\tilde{a})=0$.

If $A(\tilde{a}\preceq\tilde{b})=0$ and $A(\tilde{b}\preceq\tilde{a})=0$, then we say that the interval numbers $\tilde{a}$ and $\tilde{b}$ are equivalent (non-inferior to each other) and we denote it by $\tilde{a}\approx\tilde{b}$. Also if $A(\tilde{a}\preceq\tilde{b})\geq0$, then $\tilde{a}\preceq\tilde{b}$ and if $A(\tilde{b}\preceq\tilde{a})\geq0$, then $\tilde{b}\preceq\tilde{a}$.

3. MAIN RESULTS

3.1. General Interval Assignment Problem

Let there are n jobs and n persons are available with different skills. If the cost of doing j^{th} work by i^{th} person is c_{ij} .Now the problem is which work is to be assigned to whom so that the cost of completion of work will be minimum. Mathematically, we can express the problem as follows:

$$\text{Minimize } \tilde{Z}\ (\text{cost})=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ij}\,\tilde{x}_{ij};\ i=1,2,\ldots n;\ j=1,2,\ldots n$$

$$\text{where } \tilde{x}_{ij}=\begin{cases}1; & \text{if } i^{th} \text{ person is assigned } j^{th} \text{ work}\\ 0; & \text{if } i^{th}\text{person is not assigned the}\\ & j^{th} \text{ work with the restrictions}\end{cases}$$

$$\sum_{i=1}^{n}\tilde{x}_{ij}=1;\ j=1,2,\ldots n., \qquad (1)$$

i.e., i^{th} person will do only one work.

$$\sum_{j=1}^{n}\tilde{x}_{ij}=1;\ i=1,2,\ldots n., \qquad (2)$$

i.e., j^{th} work will be done only by one person.

3.2. Interval Hungarian Method

In this section an algorithm to solve assignment problem with generalized interval arithmetic using Hungarian method:

Step 1 Find out the mid values of each interval in the cost matrix.

Step 2 Subtract the interval which have smallest mid value in each row from all the entries of its row.

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Step 3 Subtract the interval which have smallest mid value from those columns which have no intervals contain zero from all the entries of its column.

Step 4 Draw lines through appropriate rows and columns so that all the intervals contain zero of the cost matrix are covered and the minimum number of such lines is used.

Step 5 Test for optimality (i) If the minimum number of covering lines is equal to the order of the cost matrix, then optimality is reached. (ii) If the minimum number of covering lines is less than the order of the matrix, then go to step 6.

Step 6 Determine the smallest mid value of the intervals which are not covered by any lines .Subtract this entry from all un-crossed element elements and add it to the crossing having an interval contain zero. Then go to step 4.

3.3. Tabular form of the Problem

The cost matrix of the interval assignment problem is given in the table below:

Table 1. Cost matrix of the interval assignment problem

Persons / Jobs	1	2	3	…..j	…..n
1	$\tilde{C}_{11}$	$\tilde{C}_{12}$	$\tilde{C}_{13}$	….. $\tilde{C}_{1j}$	….. $\tilde{C}_{1n}$
2	$\tilde{C}_{21}$	$\tilde{C}_{22}$	$\tilde{C}_{23}$	….. $\tilde{C}_{2j}$	….. $\tilde{C}_{2n}$
. . . i	. . . $\tilde{C}_{i1}$	. . . $\tilde{C}_{i2}$	. . . $\tilde{C}_{i3}$	. . . $\tilde{C}_{ij}$	. . . $\tilde{C}_{in}$
. . . n	. . . $\tilde{C}_{n1}$	. . . $\tilde{C}_{n2}$	. . . $\tilde{C}_{n3}$	……. ……. ……. …… $\tilde{C}_{nj}$	…… …… …… …… $\tilde{C}_{nn}$

4. Numerical Examples

Example 4.1

Let us consider an assignment problem discussed by Sarangam Majumda [23]. The assignment cost of assigning any operator to any one machine is given in the following table.

Table 2. Cost matrix with crisp entries

	I	II	III	IV
A	10	5	13	15
B	3	9	18	3
C	10	7	3	2
D	5	11	9	7

By applying Hungarian method, Sarangam Majumdar got an optimal assignment as A, B, C, D machines are assign to II, IV, III, I operators respectively and minimum assignment cost is 16.
They converted this crisp assignment problem as an interval assignment problem as given below. Now the cost matrix of the interval assignment problem is

Table 3. Cost matrix with interval entries

	I	II	III	IV
A	[9,11]	[4,6]	[12,14]	[14,16]
B	[2,4]	[8,10]	[17,19]	[2,4]
C	[9,11]	[6,8]	[2,4]	[1,3]
D	[4,6]	[10,12]	[8,10]	[6,8]

Applying their interval Hungarian method, Sarangam Majumdar obtained the optimal assignment as A, B, C, D machines are assign to II, I, III, IV operators respectively and the optimum assignment cost as [14, 22]. After stating that the above assignment is optimal, they claim that the solution is not unique and another optimal assignment can be obtained as A, B, C, D are assign to II, IV, III, I respectively and minimum assignment cost is [12 , 20]. Hence their result and their conclusion violate the concept of optimality.

Now we shall solve the same interval assignment problem given in table 3 by applying the method proposed in this paper.

Let us express all the interval parameters $\tilde{a}=[a_1, a_2]$ in terms of midpoint and width as $\tilde{a}=[a_1,a_2]=\langle m(\tilde{a}), w(\tilde{a})\rangle$.
Now the given interval transportation problem becomes

Table 4. Cost matrix with interval entries

	I	II	III	IV
A	<10,1>	<5,1>	<13,1>	<15,1>
B	<3,1>	<9,1>	<18,1>	<3,1>
C	<10,1>	<7,1>	<3,1>	<2,1>
D	<5,1>	<1,1>	<9,1>	<7,1>

Table 5. Cost matrix with interval entries

	I	II	III	IV
A	<5,1>	<0,1>	<8,1>	<10,1>
B	<0,1>	<6,1>	<15,1>	<0,1>
C	<8,1>	<5,1>	<1,1>	<0,1>
D	<4,1>	<0,1>	<8,1>	<6,1>

Table 6. Cost matrix with interval entries

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	I	II	III	IV
A	<5,1>	<0,1>	<7,1>	<10,1>
B	<0,1>	<6,1>	<14,1>	<0,1>
C	<8,1>	<5,1>	<0,1>	<0,1>
D	<4,1>	<0,1>	<7,1>	<6,1>

Table 7. Cost matrix with interval entries

	I	II	III	IV
A	<5,1>	<0,1>	<7,1>	<10,1>
B	<0,1>	<6,1>	<14,1>	<0,1>
C	<8,1>	<5,1>	<0,1>	<0,1>
D	<4,1>	<0,1>	<7,1>	<6,1>

Table 8. Cost matrix with interval entries

	I	II	III	IV
A	<1,1>	<0,1>	<3,1>	<6,1>
B	<0,1>	<10,1>	<14,1>	<0,1>
C	<8,1>	<9,1>	<0,1>	<0,1>
D	<0,1>	<0,1>	<3,1>	<2,1>

Table 9. Cost matrix with interval entries

	I	II	III	IV
A	<10,1>	**<5,1>**	<13,1>	<15,1>
B	<3,1>	<9,1>	<18,1>	**<3,1>**
C	<10,1>	<7,1>	**<3,1>**	<2,1>
D	**<5,1>**	<1,1>	<9,1>	<7,1>

The optimal assignment schedule is given by

$A \to II,\ B \to IV,\ C \to III,\ D \to I$

The optimal assignment cost

= <5, 1> + <3, 1> + <3, 1> + <5, 1>

= <16, 1>

= [15, 17]

It is to be noted that our solution [15, 17] is very much sharper than the solution [12, 20] obtained by Sarangam Majumdar

Example 4.2

Let us consider an interval assignment problem with rows representing 4 areas A, B, C, D and columns representing the salesmen I, II,III, IV. The cost matrix (C_{ij}) is given whose elements are interval numbers. The problem is to find the optimal assignment so that the total cost of area assignment becomes minimum.

Table 10. Cost matrix with interval entries

	I	II	III	IV
A	[2,4]	[4,6]	[6,8]	[0,2]
B	[8,10]	[7,9]	[11,13]	[9,11]
C	[12,14]	[7,8]	[13,15]	[1,3]
D	[4,6]	[6,8]	[9,11]	[5,7]

Let us express all the interval parameters $\tilde{a} = [a_1, a_2]$ in terms of midpoint and width as $\tilde{a} = [a_1, a_2] = \langle m(\tilde{a}), w(\tilde{a}) \rangle$.

Now the given interval transportation problem becomes

Table 11. Cost matrix with interval entries

	I	II	III	IV
A	<3,1>	<5,1>	<7,1>	<1,1>
B	<9,1>	<8,1>	<12,1>	<10,1>
C	<13,1>	<8,1>	<14,1>	<2,1>
D	<5,1>	<7,1>	<10,1>	<6,1>

Table 12. Cost matrix with interval entries

	I	II	III	IV
A	<2,1>	<4,1>	<6,1>	<0,1>
B	<1,1>	<0,1>	<4,1>	<2,1>
C	<11,1>	<6,1>	<12,1>	<0,1>
D	<0,1>	<2,1>	<5,1>	<1,1>

Table 13. Cost matrix with interval entries

	I	II	III	IV
A	<2,1>	<4,1>	<2,1>	<0,1>
B	<1,1>	<0,1>	<0,1>	<2,1>
C	<11,1>	<6,1>	<8,1>	<0,1>
D	<0,1>	<2,1>	<1,1>	<1,1>

Table 14. Cost matrix with interval entries

	I	II	III	IV
A	<2,1>	<4,1>	<2,1>	<0,1>
B	<1,1>	<0,1>	<0,1>	<2,1>
C	<11,1>	<6,1>	<8,1>	<0,1>
D	<0,1>	<2,1>	<1,1>	<1,1>

Table 11. Cost matrix with interval entries

	I	II	III	IV
A	<0,1>	<2,1>	**<0,1>**	<0,1>
B	<1,1>	**<0,1>**	<0,1>	<4,1>
C	<9,1>	<4,1>	<6,1>	**<0,1>**
D	**<0,1>**	<2,1>	<1,1>	<3,1>

The optimal assignment schedule is given by

$A \to III,\ B \to II,\ C \to IV,\ D \to I$

The optimal assignment cost

= <7, 1> + <8, 1> + <2, 1> + <5, 1>

= <22, 1>

= [21, 23]

5. Conclusion

In this paper, a new approach to solve the assignment problems with generalized interval arithmetic is proposed. To show the efficacy of the proposed method numerical examples are solved and corresponding results are compared.

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